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GREEN FUNCTIONS OF PIEZOELECTRIC MATERIAL WITH AN ELLIPTIC HOLE OR INCLUSION

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Abstract—In this paper, the Green functions of infinite two-dimensional piezoelectric material containing an elliptic hole or an elliptic piezoelectric inclusion are obtained with the extended Stroh formalism. The results can be used to investigate the mechanical–electric coupling behaviours of infinite piezoelectric material with hole or inclusion acted by a set of generalized concentrated forces, and more importantly as kernels of boundary integral equations in BEM analyses for finite size piezoelectric materials with general forces and boundary conditions. © 1997 Elsevier Science Ltd

1. INTRODUCTION

The problems of piezoelectric materials with defects, such as dislocations, inclusions, holes and cracks, have received considerable attention due to the practical importance of such problems [for example, Pak (1992); Suo *et al.* (1992); Dascalu and Maugin (1995) for fracture problems; Wang (1992); Liang *et al.* (1995) for inclusion problems]. These studies present a good understanding of mechanical and electric behaviours of piezoelectric materials or structures with defects, and are helpful to the material and structural designs. However, most of the analytical solutions presented in the literature are often restricted to some special loading and boundary conditions, such as uniform loading and infinite boundary conditions etc., while the problems encountered in engineering are always finite size subjected to general mechanical-electric loading, and no analytical solutions could be found in these cases. Therefore, it is very necessary to develop efficient computational procedures suitable for general piezoelectric problems. The boundary element method has been considered to be a good alternative for treating piezoelectric boundary value problems. Some efforts, indeed, have been made in this direction, which can be found in recent works by Lu and Mahrenholtz (1994a), Lee and Jiang (1994), Khutoryansky and Sosa (1995) etc.

It is well-known that Green functions, or fundamental solutions, are a most important part of boundary element analysis. For boundary value problems with holes or inclusions, if special Green functions satisfying the hole or inclusion boundary conditions could be found, the integrals around the hole or inclusion surfaces could, therefore, be avoided, which could save a lot of computing time and improve numerical results. The idea of this treatment has been widely used in the analysis of solid mechanics with holes or cracks [Mukherjee (1982); Hwu and Yen (1991); Lu *et al.* (1991); Lu *et al.* (1992)]. For piezoelectric problems, however, the studies of corresponding fundamental solutions satisfying special boundary conditions have not caused enough attention, according to the literature, although the fundamental solutions of homogeneous media for both static and dynamic situations have been studied by many researchers [for example, Lee and Jiang (1994); Norris (1994); Khutoryansky and Sosa (1995) etc.]. Part of the reason comes from mathematical difficulties in treating both mechanical–electric coupling and material anisotropy.

In this paper, generalized plane problems of piezoelectric materials are considered, with special concentration on finding Green functions of piezoelectric bodies with holes or inclusions, which is an important aspect of BEM formulation for solving the problems. To do so, extended Stroh formalism is used. As is well-known, Stroh formalism is a powerful and elegant method of treating generalized plane strain problems of anisotropic elasticity. Many useful results have been obtained according to this formalism [see, for example, Ting (1991)]. By careful modification, it can also be extended to solve other problems, such as bending of anisotropic elastic plates [Lu and Mahrenholtz (1994b); Lu and Wu (1996)]. Extended Stroh formalism has also found many applications to piezoelectric problems [Suo *et al.* (1995); Dascalu and Maugin (1995); Liang *et al.* (1995), Fan *et al.* (1996)]. Actually, the Stroh formalism for anisotropic elasticity could be considered to be a limit situation of the extended Stroh formalism for piezoelectric material when mechanical and electric effects are no longer coupled with each other. This understanding can help us to extend many useful results concerning anisotropic elasticity obtained by Stroh formalism to piezoelectric problems.

In the first part of this article, basic relations of piezoelectric problems are listed, and extended Stroh formalism for piezoelectric problems is introduced with necessary derivations, which is rarely shown in literature. The authors feel this knowledge is useful to help us understand why and how those important notations are introduced, and when necessary modify the formalism to solve other relative problems.

With the extended Stroh formalism, the problems of infinite two-dimensional piezoelectric material containing an elliptic hole and an elliptic piezoelectric inclusion, acted upon by a set of generalized concentrated forces, are considered. The solutions of these problems are the Green functions, particularly those satisfying the boundary conditions of holes or inclusions in BEM analyses. The merits of this kind of modified Green function have been pointed out above.

2. BASIC EQUATIONS AND STROH FORMALISM

In a fixed rectangular co-ordinate system x_i ($i = 1, 2, 3$), the linear constitutive relations for piezoelectric materials are given by (Maugin, 1988)

$$\sigma_{ij} = C_{ijkl}\gamma_{kl} - e_{kij}E_k,$$
$$D_i = e_{ikl}\gamma_{kl} + \varepsilon_{ik}E_k, \tag{1}$$

where σ_{ij}, γ_{ij}, D_i and E_k are stress, strain, electric displacement and electric field components, respectively. C_{ijkl}, e_{kij} and ε_{ik} are elastic, piezoelectric and dielectric constants, respectively, which satisfy the following symmetries:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad e_{kij} = e_{kji}, \quad \varepsilon_{ik} = \varepsilon_{ki}. \tag{2}$$

If u_i are mechanical displacements and ϕ electric potential, the deformation relations are

$$\gamma_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad E_i = -\phi_{,i}. \tag{3}$$

In the absence of body forces and free charges, the governing equations of linear piezoelectricity are

$$\sigma_{ij,i} = 0, \quad D_{i,i} = 0. \tag{4}$$

Substituting eqn (3) into eqn (1) yields

$$\sigma_{ij} = C_{ijkl}u_{k,l} + e_{kji}\phi_{,k},$$
$$D_i = e_{ikl}u_{k,l} - \varepsilon_{ik}\phi_{,k}. \tag{5}$$

Furthermore, the combination of eqns (4) and (5) results in a system of four partial differential equations coupling the displacement components and electric potential, namely

$$\begin{aligned}(C_{ijkl}u_k + e_{lji}\phi)_{,li} &= 0,\\ (e_{ikl}u_k - \varepsilon_{il}\phi)_{,li} &= 0.\end{aligned} \tag{6}$$

For two-dimensional deformations in which u_k and ϕ depend on x_1 and x_2 only, a general solution to eqn (6) can be written as

$$\mathbf{u} = \{u_k, \phi\}^{\mathrm{T}} = \mathbf{a}f(z), \quad z = x_1 + px_2, \tag{7}$$

or

$$u_k = a_k f(z), \quad \phi = a_4 f(z), \tag{8}$$

in which p and a are constants to be determined, and $f(z)$ is an arbitrary function of z. Substituting eqn (7) or (8) into eqn (6) gives

$$\begin{aligned}&[C_{1jk1} + p(C_{2jk1} + C_{1jk2}) + p^2 C_{2jk2}]a_k + [e_{1j1} + p(e_{1j2} + e_{2j1}) + p^2 e_{2j2}]a_4 = 0,\\ &[e_{1k1} + p(e_{1k2} + e_{2k1}) + p^2 e_{2k2}]a_k - [\varepsilon_{11} + p(\varepsilon_{12} + \varepsilon_{21}) + p^2 \varepsilon_{22}]a_4 = 0.\end{aligned} \tag{9}$$

Since the symmetric relations (2) exist, the above equations can be written in matrix notation as

$$\{\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T}\}\mathbf{a} = \mathbf{0}, \tag{10}$$

where

$$\mathbf{Q} = \begin{bmatrix} C_{1jk1} & e_{1j1} \\ e_{1k1}^{\mathrm{T}} & -\varepsilon_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} C_{1jk2} & e_{1j2} \\ e_{2k1}^{\mathrm{T}} & -\varepsilon_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} C_{2jk2} & e_{2j2} \\ e_{2k2}^{\mathrm{T}} & -\varepsilon_{22} \end{bmatrix}, \tag{11}$$

in which the superscript T indicates transposition. The 4×4 matrices $\mathbf{Q}$ and $\mathbf{T}$ are symmetric. To express stresses and electric displacements with the general solutions, inserting eqn (8) into eqn (5) gives

$$\begin{aligned}\sigma_{ij} &= [(C_{ijk1} + pC_{ijk2})a_k + (e_{1ji} + pe_{2ji})a_4]f'(z),\\ D_i &= [(e_{ik1} + pe_{ik2})a_k - (\varepsilon_{i1} + p\varepsilon_{i2})a_4]f'(z),\end{aligned} \tag{12}$$

or in matrix notation as

$$\begin{Bmatrix} \sigma_{2j} \\ D_2 \end{Bmatrix} = (\mathbf{R}^{\mathrm{T}} + p\mathbf{T})\mathbf{a}f'(z), \quad \begin{Bmatrix} \sigma_{1j} \\ D_1 \end{Bmatrix} = (\mathbf{Q} + p\mathbf{R})\mathbf{a}f'(z), \tag{13}$$

where the matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are given in eqn (11). Defining

$$\mathbf{b} = (\mathbf{R}^{\mathrm{T}} + p\mathbf{T})\mathbf{a}, \tag{14}$$

and comparing it with eqn (10), the following relation exists:

$$\mathbf{b} = (\mathbf{R}^{\mathrm{T}} + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}. \tag{15}$$

By introducing another solution

$$\Psi = \mathbf{b}f(z), \tag{16}$$

the formulas (13) can then be expressed in a simple form as

$$\begin{Bmatrix} \sigma_{2j} \\ D_2 \end{Bmatrix} = \Psi_{,1}, \quad \begin{Bmatrix} \sigma_{1j} \\ D_1 \end{Bmatrix} = -\Psi_{,2}. \tag{17}$$

It is known that the four-order eqn (10) is an eigenvalue problem, which gives four pairs of complex conjugates and corresponding vectors. If p_α ($\alpha = 1, 2, 3, 4$) are the eigenvalues with positive imaginary part, and $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ the associated vectors, it can be defined that

$$p_{\alpha+4} = \bar{p}_\alpha, \quad \mathbf{a}_{\alpha+4} = \bar{\mathbf{a}}_\alpha, \quad \mathbf{b}_{\alpha+4} = \bar{\mathbf{b}}_\alpha, \quad (\alpha = 1, 2, 3, 4), \tag{18}$$

where the overbar denotes the complex conjugate. Assuming that p_α are distinct, the general solutions for $\mathbf{u}$ and Ψ, therefore, can be expressed as

$$\mathbf{u} = \sum_{\alpha=1}^{4} \{\mathbf{a}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{a}}_\alpha f_{\alpha+4}(\bar{z}_\alpha)\},$$

$$\Psi = \sum_{\alpha=1}^{4} \{\mathbf{b}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{b}}_\alpha f_{\alpha+4}(\bar{z}_\alpha)\}, \tag{19}$$

where $f_1, f_2, \ldots, f_8$ are arbitrary functions of their arguments and z_α are given by

$$z_\alpha = x_1 + p_\alpha x_2. \tag{20}$$

For a given boundary value problem, the functions f_α ($\alpha = 1, 2, \ldots, 8$) are sought to satisfy the proper boundary conditions. In many applications, f_α could be assumed to have the same function form, i.e.

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha), \quad f_{\alpha+4}(\bar{z}_\alpha) = \bar{q}_\alpha \bar{f}(\bar{z}_\alpha), \quad (\alpha = 1, 2, 3, 4), \tag{21}$$

where q_α are complex constants to be determined. Defining

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4], \tag{22}$$

and

$$\langle f(z_*) \rangle = \mathrm{diag}\,[f(z_1), \; f(z_2), \; f(z_3), \; f(z_4)],$$

$$\mathbf{q} = \{q_1, q_2, q_3, q_4\}^{\mathrm{T}}, \tag{23}$$

eqn (19) can then be written, by considering relations (21), as

$$\mathbf{u} = 2\,\mathrm{Re}\,\{\mathbf{A}\langle f(z_*)\rangle \mathbf{q}\}, \quad \Psi = 2\,\mathrm{Re}\,\{\mathbf{B}\langle f(z_*)\rangle \mathbf{q}\}. \tag{24}$$

Hereafter, the angular bracket $\langle \cdot \rangle$ always stands for diagonal matrix.

So far, the extended Stroh formalism for two-dimensional piezoelectricity has been introduced following a logical order with necessary derivations. It can be seen that the Stroh formalism for piezoelectricity has the same form and common properties, in most situations, as that for anisotropic electricity. Actually, in the case of uncoupling between elastic and electric fields, i.e. $e_{kij} = 0$ in eqn (1), the above formulas are reduced to ones in anisotropic elastic mechanics. Therefore, most properties and identities existing in the Stroh formalism for anisotropic electricity can be extended for solving piezoelectric problems. A few important relations which extend from anisotropic elasticity, and which will be used later, are listed here omitting demonstration for simplicity.

According to relation (15), the eigenvalue problem (10) can be expressed in a standard form as

$$\mathbf{N}\xi = p\xi, \tag{25}$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \quad \xi = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}, \tag{26}$$

and

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}. \tag{27}$$

From eqn (25), the following orthogonality relations among the eigenvectors can be obtained:

$$\begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \bar{\mathbf{B}}^T & \bar{\mathbf{A}}_T \end{bmatrix} = [\mathbf{I}], \tag{28}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the eigenmatrices given by eqn (22). Expansion of eqn (28) gives

$$\mathbf{A}\mathbf{A}^T + \bar{\mathbf{A}}\bar{\mathbf{A}}^T = \mathbf{B}\mathbf{B}^T + \bar{\mathbf{B}}\bar{\mathbf{B}}^T = \mathbf{0}, \quad \mathbf{B}\mathbf{A}^T + \bar{\mathbf{B}}\bar{\mathbf{A}}^T = \mathbf{A}\mathbf{B}^T + \bar{\mathbf{A}}\bar{\mathbf{B}}^T = \mathbf{I}, \tag{29}$$

from which the real matrices $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{L}$ can be defined by (Ting, 1991)

$$\mathbf{S} = i(2\mathbf{A}\mathbf{B}^T - \mathbf{I}), \quad \mathbf{H} = i2\mathbf{A}\mathbf{A}^T, \quad \mathbf{L} = -i2\mathbf{B}\mathbf{B}^T, \tag{30}$$

and the following relations exist

$$\mathbf{H}\mathbf{L} - \mathbf{S}\mathbf{S} = \mathbf{L}\mathbf{H} - \mathbf{S}^T\mathbf{S}^T = \mathbf{I}, \quad \mathbf{L}\mathbf{S} + \mathbf{S}^T\mathbf{L} = \mathbf{S}\mathbf{H} + \mathbf{H}\mathbf{S}^T = \mathbf{0}. \tag{31}$$

3. GREEN'S FUNCTIONS OF PIEZOELECTRIC MATERIAL WITH AN ELLIPTIC HOLE

In this section, the problem of an infinite piezoelectric material with an elliptic hole (and, in the limit, a crack) under a concentrated mechanical force and electric charge density vector $\mathbf{p}$ applied at a point $\mathbf{x}^* = (x_1^*, x_2^*)$, as shown in Fig. 1, is considered.

It is a relatively simple, but a rather important problem, which can find many applications. For example, the solution of the problem, i.e. Green's function, can be used as kernels of boundary element method for solving the problem of finite size piezoelectric material containing an elliptic hole or crack.

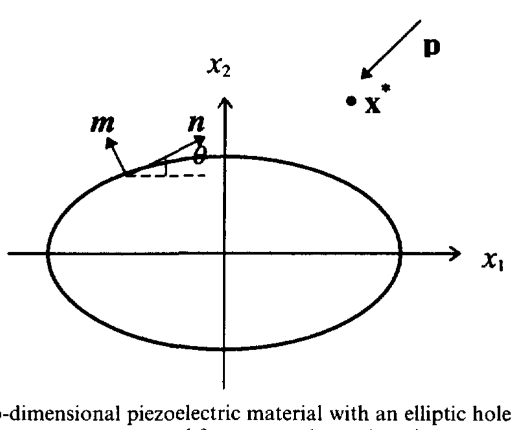

Fig. 1. An infinite two-dimensional piezoelectric material with an elliptic hole or inclusion under a concentrated force $\mathbf{p}$ acted at point $\mathbf{x}^*$.

3.1. *Boundary conditions and general solutions*

The contour Γ of the elliptic hole is described by

$$x_1 = a\cos\vartheta, \quad x_2 = b\sin\vartheta, \tag{32}$$

where ϑ is a real parameter, and a and b are the semi-major and the semi-minor axes of the ellipse, respectively. Let **n** and **m** be the unit vectors tangential and normal to Γ which are given by

$$\mathbf{n}^{\mathrm{T}} = (\cos\theta, \sin\theta, 0), \quad \mathbf{m}^{\mathrm{T}} = (-\sin\theta, \cos\theta, 0), \tag{33}$$

where θ is directed counterclockwise from the positive x_1-axis to the direction of **n**, as shown in Fig. 1. From eqn (17), the traction and the electric displacement components along the hole surface can be expressed as

$$\mathbf{t}_m = \{\boldsymbol{\sigma}\cdot\mathbf{m}, \mathbf{D}\cdot\mathbf{m}\}^{\mathrm{T}} = \boldsymbol{\Psi}_{,n}, \tag{34}$$

where $\boldsymbol{\sigma}$ and $\mathbf{D}$ are given by eqn (5), and $\mathbf{t}_m$ is defined as generalized traction vector. If it is assumed to be traction and electric charge free along the hole surface, the physical and boundary conditions of the considered problem can be written as

$$\begin{aligned}
&\mathbf{t}_m = \boldsymbol{\Psi}_{,n} = \mathbf{0}, \quad \text{on } \Gamma,\\
&\oint_C \mathrm{d}\boldsymbol{\Psi} = \mathbf{p}, \text{ for any closed curver } C \text{ enclosing the point } \mathbf{x}^*,\\
&\oint_C \mathrm{d}\mathbf{u} = \mathbf{0}, \text{ for any closed curver } C \text{ enclosing the point } \mathbf{x}^*,\\
&\boldsymbol{\sigma}_{ij} \to 0, \quad D_i \to 0, \quad \text{at infinity.}
\end{aligned} \tag{35}$$

In solving the problem, proper functions $f_\alpha(z_\alpha)$ in general solution eqn (24) should be chosen to satisfy the conditions (35).

Consider the mapping

$$z_\alpha = \frac{1}{2}\left\{(a - ibp_\alpha)\zeta_\alpha + (a + ibp_\alpha)\frac{1}{\zeta_\alpha}\right\}, \tag{36}$$

which transforms the region outside the elliptic hole in the z_α-plane onto the exterior of a unit circle in the ζ_α-plane. Since the roots of $\mathrm{d}z_\alpha/\mathrm{d}\zeta_\alpha = 0$, i.e. $\pm\sqrt{a+ibp_\alpha/a-ibp_\alpha}$ with $\mathrm{Im}\{p_\alpha\} > 0$, are located inside the unit circle $|\zeta_\alpha| = 1$, the transformation (36) is one-to-one outside the hole with $\zeta_\alpha \to \infty$ as $z_\alpha \to \infty$, and $\zeta_\alpha|_\Gamma = e^{i\vartheta}$ when z_α is on the hole boundary.

By considering the conditions (35) and the transformation (36), the general solution for the problem can be written as (Hwu and Yen, 1991)

$$\begin{aligned}
\mathbf{u} &= 2\,\mathrm{Re}\,\{\mathbf{A}\langle f_0(z_*)\rangle\mathbf{q}_0\} + 2\sum_{\beta=1}^{4}\mathrm{Re}\,\{\mathbf{A}\langle f_\beta(z_*)\rangle\mathbf{q}_\beta\},\\
\boldsymbol{\Psi} &= 2\,\mathrm{Re}\,\{\mathbf{B}\langle f_0(z_*)\rangle\mathbf{q}_0\} + 2\sum_{\beta=1}^{4}\mathrm{Re}\,\{\mathbf{B}\langle f_\beta(z_*)\rangle\mathbf{q}_\beta\},
\end{aligned} \tag{37}$$

where

$$\langle f_k(z_*)\rangle = \mathrm{diag}\,[f_k(z_1),\ f_k(z_2),\ f_k(z_3),\ f_k(z_4)], \quad k = 1,2,3,4 \tag{38}$$

and

$$
\begin{aligned}
f_0(z_\alpha) &= \ln(\zeta_\alpha - \zeta_\alpha^*),\\
f_\beta(z_\alpha) &= \ln(\zeta_\alpha^{-1} - \bar{\zeta}_\beta^*), \quad \beta = 1,2,3,4,\\
\zeta_\alpha &= \frac{z_\alpha + \sqrt{z_\alpha^2 - a^2 - p_\alpha^2 b^2}}{a - ip_\alpha b},\\
\zeta_\alpha^* &= \frac{z_\alpha^* + \sqrt{z_\alpha^{*2} - a^2 - p_\alpha^2 b^2}}{a - ip_\alpha b},\\
z_\alpha^* &= x_1^* + p_\alpha x_2^*,
\end{aligned}
\tag{39}
$$

where $\alpha = 1, 2, 3, 4$, respectively, $\mathbf{q}_0$ and $\mathbf{q}_\beta$ are four-order unknown constant vectors to be determined by satisfying the conditions (35). In eqn (37), the terms concerning functions $f_0(z_\alpha)$ are chosen in order to satisfy conditions $(35)_2$, which are just Green's functions of an infinite piezoelectric material without hole, while the analytical solutions $f_\beta(z_\alpha)$, outside the ellipse, are added to modify the solutions due to the existence of the traction-free hole. Now, the problem is reduced to determine unknowns $\mathbf{q}_0$ and $\mathbf{q}_\beta$.

3.2. *Determination of constant vectors* $\mathbf{q}_0$ *and* $\mathbf{q}_\beta$

To determine the unknown constants, $\mathbf{\Psi}_{,n}$ along the elliptic boundary should be calculated. For this reason, a few relations along the hole are given first. Let $\rho\,\mathrm{d}\vartheta$ be the infinitesimal arclength of the hole boundary Γ where

$$
\rho = (a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta)^{1/2}. \tag{40}
$$

From eqns (32) and (33), we have

$$
\cos\theta = \frac{-\mathrm{d}x_1}{\rho\,\mathrm{d}\vartheta} = \frac{a}{\rho}\sin\vartheta, \quad \sin\theta = \frac{-\mathrm{d}x_2}{\rho\,\mathrm{d}\vartheta} = -\frac{b}{\rho}\cos\vartheta. \tag{41}
$$

Differentiating both sides of eqn (20) and evaluating the results along the hole boundary by using eqn (41) gives

$$
\mathrm{d}z_\alpha|_\Gamma = -\rho(\cos\theta + p_\alpha \sin\theta)\,\mathrm{d}\vartheta. \tag{42}
$$

On the hole boundary, we also have

$$
\begin{aligned}
\mathrm{d}\zeta_\alpha|_\Gamma &= (-\sin\vartheta + i\cos\vartheta)\,\mathrm{d}\vartheta = ie^{i\vartheta}\,\mathrm{d}\vartheta,\\
\frac{\partial}{\partial n}f(z_\alpha) &= \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial n} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial n} = (\cos\theta + p_\alpha \sin\theta)\frac{\mathrm{d}f}{\mathrm{d}z_\alpha}.
\end{aligned}
\tag{43}
$$

By use of the relations (40)–(43) in eqns (38) and (39), we obtain

$$
\begin{aligned}
\frac{\partial}{\partial n}\langle f_0(z_*)\rangle|_\Gamma &= \mathrm{diag}\,[c_1, c_2, c_3, c_4] = \sum_{\beta=1}^{4} c_\beta \mathbf{I}_\beta,\\
\frac{\partial}{\partial n}\langle f_\beta(z_*)\rangle|_\Gamma &= \bar{c}_\beta \mathbf{I}, \quad \beta = 1,2,3,4,
\end{aligned}
\tag{44}
$$

where

$$
c_\beta = \frac{-ie^{i\vartheta}}{\rho(e^{i\vartheta} - \zeta_\beta^*)}, \tag{45}
$$

and

$$\mathbf{I}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{I}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{I}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{I}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{46}$$

$\mathbf{I}$ is 4×4 unit matrix. Substituting eqns (44) and $(37)_2$ into eqn $(35)_1$ gives

$$2 \sum_{\beta=1}^{4} \operatorname{Re} \{c_\beta \mathbf{B} \mathbf{I}_\beta \mathbf{q}_0 + \bar{c}_\beta \mathbf{B} \mathbf{I} \mathbf{q}_\beta\} = \mathbf{0}. \tag{47}$$

Similarly, we have from eqn $(35)_2$ and $(35)_3$

$$\mathbf{B}\mathbf{q}_0 - \bar{\mathbf{B}}\bar{\mathbf{q}}_0 = \frac{1}{2\pi i}\mathbf{p},$$

$$\mathbf{A}\mathbf{q}_0 - \bar{\mathbf{A}}\bar{\mathbf{q}}_0 = \mathbf{0}, \tag{48}$$

from which $\mathbf{q}_0$ can be obtained by considering the orthogonality relations (28), i.e.

$$\mathbf{q}_0 = \frac{1}{2\pi i}\mathbf{A}^{\mathrm{T}}\mathbf{p}. \tag{49}$$

After $\mathbf{q}_0$ is known, a solution of $\mathbf{q}_\beta$ could be obtained from eqn (47) given by

$$\mathbf{q}_\beta = -\mathbf{B}^{-1}\bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{q}}_0, \quad \beta = 1, 2, 3, 4. \tag{50}$$

Substituting eqns (49) and (50) into eqn (37), the Green's function for infinite piezoelectric material with a traction-free and electric impermeability elliptic hole can be written as

$$\mathbf{u} = \frac{1}{\pi}\operatorname{Im}\{\mathbf{A}\langle f_0(z_*)\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{p} + \frac{1}{\pi}\sum_{\beta=1}^{4}\operatorname{Im}\{\mathbf{A}\langle f_\beta(z_*)\rangle\mathbf{B}^{-1}\bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{A}}^{\mathrm{T}}\}\mathbf{p},$$

$$\Psi = \frac{1}{\pi}\operatorname{Im}\{\mathbf{B}\langle f_0(z_*)\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{p} + \frac{1}{\pi}\sum_{\beta=1}^{4}\operatorname{Im}\{\mathbf{B}\langle f_\beta(z_*)\rangle\mathbf{B}^{-1}\bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{A}}^{\mathrm{T}}\}\mathbf{p}. \tag{51}$$

As pointed out before, the terms concerning $\langle f_0(z_*)\rangle$ in eqn (51) are Green's functions of homogeneous media, which will be used in the next section.

3.3. *Internal stresses and electric displacements*

After Green's function is obtained, the effect caused by the concentrated forces could be investigated by considering the variations of internal stresses and electric displacements as well as the generalized hoop stresses along the elliptic boundary.

From eqn (39), we have

$$\frac{\partial f_0(z_\alpha)}{\partial x_1}=\frac{\zeta_\alpha}{\partial_\alpha-\zeta_\alpha^*}\frac{1}{\sqrt{z_\alpha^{*}-a^2-p_\alpha^2b^2}},\quad \frac{\partial f_0(z_\alpha)}{\partial x_2}=p_\alpha\frac{\partial f_0(z_\alpha)}{\partial x_1},$$
$$\frac{\partial f_\beta(z_\alpha)}{\partial x_1}=\frac{1}{\zeta_\alpha\bar{\zeta}_\beta^*-1}\frac{1}{\sqrt{z_\alpha^{*}-a^2-p_\alpha^2b^2}},\quad \frac{\partial f_\beta(z_\alpha)}{\partial x_2}=p_\alpha\frac{\partial f_\beta(z_\alpha)}{\partial x_1},\tag{52}$$

and on the hole boundary, eqn (52) is reduced to

$$\left.\frac{\partial f_0(z_\alpha)}{\partial x_1}\right|_\Gamma=\frac{c_\alpha}{\cos\theta+p_\alpha\sin\theta},\quad \left.\frac{\partial f_\beta(z_\alpha)}{\partial x_1}\right|_\Gamma=\frac{\bar{c}_\beta}{\cos\theta+p_\alpha\sin\theta}.\tag{53}$$

Substituting eqns (52) and $(51)_2$ into eqn (17), the internal stresses and the electric displacements at any point can be expressed as

$$\begin{Bmatrix}\sigma_{2j}\\D_2\end{Bmatrix}=\Psi_{,1}=\frac{1}{\pi}\mathrm{Im}\,\{\mathbf{B}\langle f_0(z_*)\rangle_{,1}\mathbf{A}^{\mathrm{T}}\}\mathbf{p}+\frac{1}{\pi}\sum_{\beta=1}^{4}\mathrm{Im}\,\{\mathbf{B}\langle f_\beta(z_*)\rangle_{,1}\mathbf{B}^{-1}\bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{A}}^{\mathrm{T}}\}\mathbf{p},$$
$$\begin{Bmatrix}\sigma_{1j}\\D_1\end{Bmatrix}=-\Psi_{,2}=-p_\alpha\begin{Bmatrix}\sigma_{2j}\\D_2\end{Bmatrix},\tag{54}$$

and the generalized hoop stresses can be calculated by the following expressions:

$$\sigma_{nn}=\sigma_{11}\cos^2\theta+\sigma_{22}\sin^2\theta+2\sigma_{12}\cos\theta\sin\theta,$$
$$\sigma_{nm}=(\sigma_{22}-\sigma_{11})\cos\theta\sin\theta+\sigma_{12}(\cos^2\theta+\sin^2\theta),$$
$$D_n=D_1\cos\theta+D_2\sin\theta,\tag{55}$$

in which σ_{ij} and D_i are calculated at the hole boundary.

3.4. *Crack and intensity factors*

By letting $b\to 0$ in eqn (32), the problem discussed before becomes an infinite piezoelectric plate with a Griffith crack of length $2a$. In this case, eqn (39) is reduced to

$$\zeta_\alpha=\frac{z_\alpha+\sqrt{z_\alpha^2-a^2}}{a},\quad \zeta_\alpha^*=\frac{z_\alpha^*+\sqrt{z_\alpha^{*2}-a^2}}{a},\tag{56}$$

and at the place ahead of the crack tip, i.e. $x_2=0$ and $|x_1|>a$, it gives

$$\zeta_\alpha=\frac{x_1+\sqrt{x_1^2-a^2}}{a}=\zeta,\tag{57}$$

and eqn (52) becomes the case

$$\frac{\partial f_0(z_\alpha)}{\partial x_1}=\frac{\zeta}{\zeta-\zeta_\alpha^*}\frac{1}{\sqrt{x_1^2-a^2}},\quad \frac{\partial f_\beta(z_\alpha)}{\partial x_1}=\frac{1}{\zeta\bar{\zeta}_\beta^*-1}\frac{1}{\sqrt{x_1^2-a^2}}.\tag{58}$$

We now calculate $(54)_1$ ahead of the crack tip. According to eqns (22) and $(58)_1$, the terms in the first braces of $(54)_1$ can be written as

$$\mathrm{Im}\,\{\mathbf{B}\langle f_0(z_*)\rangle_{,1}\mathbf{A}^{\mathrm{T}}\}=\frac{\zeta}{\sqrt{x_1^2-a^2}}\sum_{\beta=1}^{4}\mathrm{Im}\left(\frac{1}{\zeta-\zeta_\beta^*}\mathbf{b}_\beta\mathbf{a}_\beta^{\mathrm{T}}\right).\tag{59}$$

Similarly, the terms concerning the second braces in eqn $(54)_1$ can be written as

$$\sum_{\beta=1}^{4} \mathrm{Im}\,\{\mathbf{B}\langle f_\beta(z_*)\rangle_{,1}\mathbf{B}^{-1}\bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{A}}^{\mathrm{T}}\} = \frac{\zeta}{\sqrt{x_1^2-a^2}}\sum_{\beta=1}^{4} \mathrm{Im}\left(\frac{1}{\zeta-\zeta^2\zeta_\beta^*}\mathbf{b}_\beta\mathbf{a}_\beta^{\mathrm{T}}\right), \tag{60}$$

in which the relations

$$\bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{A}}^{\mathrm{T}} = \bar{\mathbf{b}}_\beta\bar{\mathbf{a}}_\beta^{\mathrm{T}}, \quad \mathrm{Im}\left(\frac{\bar{\mathbf{b}}_\beta\bar{\mathbf{a}}_\beta^{\mathrm{T}}}{\zeta^2\zeta_\beta^*-\zeta}\right) = \mathrm{Im}\left(\frac{\mathbf{b}_\beta\mathbf{a}_\beta^{\mathrm{T}}}{\zeta-\zeta^2\zeta_\beta^*}\right),$$

have been used. In this way, eqn $(54)_1$ can be expressed as

$$\begin{Bmatrix}\sigma_{21}\\ \sigma_{22}\\ \sigma_{23}\\ D_2\end{Bmatrix} = \frac{1}{\pi a}\left\{1+\frac{x_1}{\sqrt{x_1^2-a^2}}\right\}\sum_{\beta=1}^{4}\mathrm{Im}\left\{\left(\frac{1}{\zeta-\zeta_\beta^*}+\frac{1}{\zeta-\zeta^2\zeta_\beta^*}\right)\mathbf{b}_\beta\mathbf{a}_\beta^{\mathrm{T}}\right\}\mathbf{p}, \tag{61}$$

where ζ and ζ_β^* are given by eqns (57) and $(56)_2$, respectively. With the definition of stress and electric displacement intensity factors, we have

$$\mathbf{K} = \begin{Bmatrix}K_{\mathrm{II}}\\ K_{\mathrm{I}}\\ K_{\mathrm{III}}\\ K_{\mathrm{IV}}\end{Bmatrix} = \lim_{x_1\to a}\sqrt{2\pi(x_1-a)}\begin{Bmatrix}\sigma_{21}\\ \sigma_{22}\\ \sigma_{23}\\ D_2\end{Bmatrix} = \frac{2}{\sqrt{\pi a}}\mathrm{Im}\left\{\sum_{\beta=1}^{4}\frac{1}{1-\zeta_\beta^*}\mathbf{b}_\beta\mathbf{a}_\beta^{\mathrm{T}}\right\}\mathbf{p}$$
$$= \frac{2}{\sqrt{\pi a}}\mathrm{Im}\left\{\mathbf{B}\left\langle\frac{1}{1-\zeta_\beta^*}\right\rangle\mathbf{A}^{\mathrm{T}}\right\}\mathbf{p}, \tag{62}$$

where K_{I}, K_{II} and K_{III} are the stress intensity factors and K_{IV} is the electric displacement intensity factor (Suo *et al.*, 1992; Dascalu and Maugin, 1995).

4. GREEN'S FUNCTIONS OF PIEZOELECTRIC MATRIX WITH A PIEZOELECTRIC INCLUSION

In this section, the problem of an infinite two-dimensional piezoelectric matrix with an elliptic piezoelectric inclusion, instead of an elliptic hole discussed in Section 3, is considered. This problem is of importance in studying mechanical and electric properties of piezoelectric solids with defects.

Assuming that the matrix and the elliptic inclusion are bonded perfectly along the interface, the mechanical displacement and electric potential as well as the stress and electric displacement should be continuous across the bonded interface. According to eqns (7) and (17) (Liang *et al.*, 1995), the interface conditions can be written as

$$\mathbf{u}_1 = \mathbf{u}_2, \quad \boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_2, \quad \text{on } \Gamma, \tag{63}$$

where the subscripts 1 and 2 in this section indicate the regions of the matrix and the inclusion, respectively. If a set of generalized concentrated forces $\mathbf{p}$ are applied at a point $\mathbf{x}^*$ in the region of matrix, as shown in Fig. 1, the solution for the problem gives the concerned Green's functions. Following the treatment by Hwu and Yen (1993), the general solutions, or Green's functions, for the problem may be expressed as

$$\left.\begin{aligned}\mathbf{u}_1 &= \mathbf{A}_1[\mathbf{f}_0(\zeta_*)+\mathbf{f}_1(\zeta_*)]+\bar{\mathbf{A}}_1\overline{[\mathbf{f}_0(\zeta_*)+\mathbf{f}_1(\zeta_*)]}\\ \boldsymbol{\Psi}_1 &= \mathbf{B}_1[\mathbf{f}_0(\zeta_*)+\mathbf{f}_1(\zeta_*)]+\bar{\mathbf{B}}_1\overline{[\mathbf{f}_0(\zeta_*)+\mathbf{f}_1(\zeta_*)]}\end{aligned}\right\},\quad \zeta_\alpha\in S_1, \tag{64}$$

and

$$\left.\begin{aligned}\mathbf{u}_2 &= \mathbf{A}_2\mathbf{f}_2(\hat{\zeta}_*)+\bar{\mathbf{A}}_2\overline{\mathbf{f}_2(\hat{\zeta}_*)}\\ \boldsymbol{\Psi}_2 &= \mathbf{B}_2\mathbf{f}_2(\hat{\zeta}_*)+\bar{\mathbf{B}}_2\overline{\mathbf{f}_2(\hat{\zeta}_*)}\end{aligned}\right\},\quad \hat{\zeta}_\alpha\in S_2, \tag{65}$$

where $\hat{\zeta}_\alpha$ is the mapped point of $\hat{z}_\alpha = x_1+\hat{p}_\alpha x_2$ in which $\hat{p}_\alpha$ is the material eigenvalue of the inclusion. $\mathbf{f}_0$ represents the singular solution of homogeneous media caused by the concentrated forces $\mathbf{p}$ acted at $\mathbf{x}^*$, which has been obtained in the last section, and can be written as

$$\mathbf{f}_0(\zeta_*) = \langle f_0(\zeta_*)\rangle\mathbf{q}_0 = \frac{1}{2\pi i}\langle\ln(\zeta_\alpha-\zeta_\alpha^*)\rangle\mathbf{A}_1^{\mathrm{T}}\,\mathbf{p}, \tag{66}$$

and the relations between variables ζ_α and z_α as well as $\hat{\zeta}_\alpha$ and $\hat{z}_\alpha$ are given by eqns (36) and (39). $\mathbf{f}_1$ and $\mathbf{f}_2$ are modified functions, to be determined, due to the existence of the inclusion, and should be holomorphic in regions S_1 and S_2, respectively. Here S_1 and S_2 indicate the regions occupied by the matrix and the inclusion, respectively. According to the analytical properties of $\mathbf{f}_1$ and $\mathbf{f}_2$ as well as the restriction of the transformation discussed in the last section, S_1 is the region, in the ζ_α-plane, outside the unit circle $|\zeta_\alpha| = 1$ while S_2 is the region of the annular ring between $|\zeta_\alpha| = 1$ and $|\sqrt{a+ibp_\alpha|a-ibp_\alpha}| < 1$. Therefore, in the annular ring, $\mathbf{f}_2$ can be represented by Laurent's expansion as (Hwu and Yen, 1993)

$$\mathbf{f}_2(\hat{\zeta}_*) = \sum_{k=-\infty}^{\infty}\langle\hat{\zeta}_\alpha^{-k}\rangle\mathbf{c}_k, \tag{67}$$

where

$$\mathbf{c}_{-k} = \hat{\boldsymbol{\Gamma}}_k\mathbf{c}_k,\quad \hat{\boldsymbol{\Gamma}}_k = \left\langle\left(\frac{a+ib\hat{p}_\alpha}{a-ib\hat{p}_\alpha}\right)^k\right\rangle,\quad \langle\hat{\zeta}_\alpha^{-k}\rangle = \mathrm{diag}\,[\hat{\zeta}_1^{-k},\hat{\zeta}_2^{-k},\hat{\zeta}_3^{-k},\hat{\zeta}_4^{-k}]. \tag{68}$$

For convenience of derivation later, (67) is symbolically written as

$$\mathbf{f}_2 = \sum_{k=0}^{\infty}\mathbf{c}_k\hat{\zeta}_*^k+\sum_{k=1}^{\infty}\hat{\boldsymbol{\Gamma}}_k\mathbf{c}_k\hat{\zeta}_*^{-k}, \tag{69}$$

in which the matrix $\langle\hat{\zeta}_\alpha^{-k}\rangle$ is replaced by a general variable, $\hat{\zeta}_*^{-k}$. As soon as the solution of $\mathbf{f}_1(\zeta_*)$ and $\mathbf{f}_2(\hat{\zeta}_*)$ are obtained, a replacement should be made. Substituting eqn (69) into eqn (65), and using the condition $\boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_2$ along the interface $\zeta_\alpha|_\Gamma = e^{i\vartheta} = \sigma$, it gives

$$\begin{aligned}\mathbf{B}_1\mathbf{f}_1(\sigma)+\bar{\mathbf{B}}_1\overline{\mathbf{f}_0(\sigma)}-\sum_{k=1}^{\infty}\{\bar{\mathbf{B}}_2\mathbf{c}_k+\mathbf{B}_2\hat{\boldsymbol{\Gamma}}_k\mathbf{c}_k\}\sigma^{-k}\\ = -\bar{\mathbf{B}}_1\overline{\mathbf{f}_1(\sigma)}-\mathbf{B}_1\mathbf{f}_0(\sigma)+\sum_{k=1}^{\infty}\{\mathbf{B}_2\mathbf{c}_k+\bar{\mathbf{B}}_2\bar{\hat{\boldsymbol{\Gamma}}}_k\bar{\mathbf{c}}_k\}\sigma^k.\end{aligned} \tag{70}$$

It is known from the theory of complex analysis that if $\mathbf{f}(\zeta_*)$ is holomorphic in the region outside the unit circle S^+, $\overline{\mathbf{f}(1/\bar{\zeta}_*)}$ is holomorphic in the region inside the unit circle S^-. According to the property, we may introduce a function which is holomorphic in the entire domain including the interface boundary (Hwu and Yen, 1993), i.e.

$$\boldsymbol{\theta}(\zeta_*) = \begin{cases} \mathbf{B}_1\mathbf{f}_1(\zeta_*)+\bar{\mathbf{B}}_1\overline{\mathbf{f}_0(1/\bar{\zeta}_*)}-\sum\limits_{k=1}^{\infty}\{\bar{\mathbf{B}}_2\bar{\mathbf{c}}_k+\mathbf{B}_2\hat{\mathbf{\Gamma}}_k\mathbf{c}_k\}\zeta_*^{-k}, & \zeta_*\in S^+, \\ -\bar{\mathbf{B}}_1\overline{\mathbf{f}_1(1/\bar{\zeta}_*)}-\bar{\mathbf{B}}_1\overline{\mathbf{f}_1(1/\bar{\zeta}_*)}-\bar{\mathbf{B}}_1\mathbf{f}_0(\zeta_*)+\sum\limits_{k=1}^{\infty}\{\mathbf{B}_2\mathbf{c}_k-\bar{\mathbf{B}}_2\bar{\hat{\mathbf{\Gamma}}}_k\bar{\mathbf{c}}_k\}\zeta_*^{-k}, & \zeta_*\in S^-. \end{cases} \tag{71}$$

Since $\boldsymbol{\theta}(\zeta_*)$ is now holomorphic and single value in the whole plane including the point at infinity, it gives, by Liouville's theorem, $\boldsymbol{\theta}(\zeta_*) \equiv$ constant. However, constant function $\mathbf{f}$ corresponds to rigid-body motion which could be neglected. Therefore, $\boldsymbol{\theta}(\zeta_*) \equiv 0$ in the whole plane, and eqn (71) can be written as

$$\begin{aligned} &\sum_{k=1}^{\infty}\{\bar{\mathbf{B}}_2\bar{\mathbf{c}}_k+\mathbf{B}_2\hat{\mathbf{\Gamma}}_k\mathbf{c}_k\}\zeta_*^{-k} = \mathbf{B}_1\mathbf{f}_1(\zeta_*)+\bar{\mathbf{B}}_1\bar{\mathbf{f}}_0(1/\zeta_*), \quad \zeta_*\in S^+, \\ &\sum_{k=1}^{\infty}\{\mathbf{B}_2\mathbf{c}_k+\bar{\mathbf{B}}_2\bar{\hat{\mathbf{\Gamma}}}_k\bar{\mathbf{c}}_k\}\zeta_*^{k} = \bar{\mathbf{B}}_1\bar{\mathbf{f}}_1(1/\zeta_*)+\mathbf{B}_1\mathbf{f}_0(\zeta_*), \quad \zeta_*\in S^-. \end{aligned} \tag{72}$$

Similarly, the continuity condition $\mathbf{u}_1 = \mathbf{u}_2$ in eqn (63) gives

$$\begin{aligned} &\sum_{k=1}^{\infty}\{\bar{\mathbf{A}}_2\bar{\mathbf{c}}_k+\mathbf{A}_2\hat{\mathbf{\Gamma}}_k\mathbf{c}_k\}\zeta_*^{-k} = \mathbf{A}_1\mathbf{f}_1(\zeta_*)+\bar{\mathbf{A}}_1\bar{\mathbf{f}}_0(1/\zeta_*), \quad \zeta_*\in S^+, \\ &\sum_{k=1}^{\infty}\{\mathbf{A}_2\mathbf{c}_k+\bar{\mathbf{A}}_2\bar{\hat{\mathbf{\Gamma}}}_k\bar{\mathbf{c}}_k\}\zeta_*^{k} = \bar{\mathbf{A}}_1\bar{\mathbf{f}}_1(1/\zeta_*)+\mathbf{A}_1\mathbf{f}_0(\zeta_*), \quad \zeta_*\in S^-. \end{aligned} \tag{73}$$

Eliminating $\mathbf{f}_1(\zeta_*)$ from eqns (72) and (73) leads to

$$\mathbf{f}_0(\zeta_*) = \frac{1}{2}\sum_{k=1}^{\infty}\mathbf{A}_1^{-1}\mathbf{H}_1\{(\bar{\mathbf{M}}_1+\mathbf{M}_2)\mathbf{A}_2\mathbf{c}_k+(\bar{\mathbf{M}}_1-\bar{\mathbf{M}}_2)\bar{\mathbf{A}}_2\bar{\hat{\mathbf{\Gamma}}}_k\bar{\mathbf{c}}_k\}\zeta_*^k, \tag{74}$$

where

$$\begin{aligned} &\mathbf{M}_j = -i\mathbf{B}_j\mathbf{A}_j^{-1} = \mathbf{H}_j^{-1}(\mathbf{I}+i\mathbf{S}_j) = (\mathbf{I}+i\mathbf{S}_j^{\mathrm{T}})^{-1}\mathbf{L}_j, \\ &\mathbf{S}_j = i(2\mathbf{A}_j\mathbf{B}_j^{\mathrm{T}}-\mathbf{I}), \quad \mathbf{H}_j = 2i\mathbf{A}_j\mathbf{A}_j^{\mathrm{T}}, \quad \mathbf{L}_j = -2i\mathbf{B}_j\mathbf{B}_j^{\mathrm{T}}, \quad j = 1,2. \end{aligned} \tag{75}$$

By Taylor's series expansion of $\mathbf{f}_0(\zeta_*)$ from eqn (67) and comparing the coefficient of corresponding terms in eqn (74), the unknown constants $\mathbf{c}_k$ can be obtained, i.e.

$$\mathbf{c}_k = \{\mathbf{G}_0-\bar{\mathbf{G}}_k\bar{\mathbf{G}}_0^{-1}\mathbf{G}_k\}^{-1}\{\mathbf{t}_k-\bar{\mathbf{G}}_k\bar{\mathbf{G}}_0^{-1}\bar{\mathbf{t}}_k\}, \quad k = 1,2,\ldots,\infty, \tag{76}$$

where

$$\begin{aligned} &\mathbf{G}_0 = (\bar{\mathbf{M}}_1+\mathbf{M}_2)\mathbf{A}_2, \quad \mathbf{G}_k = (\mathbf{M}_1-\mathbf{M}_2)\mathbf{A}_2\hat{\mathbf{\Gamma}}_k, \\ &\mathbf{t}_k = -\frac{1}{2\pi}\mathbf{A}_1^{-\mathrm{T}}\left\langle -\frac{1}{k}\left(\frac{1}{\zeta_\alpha^*}\right)^k\right\rangle\mathbf{A}_1^{\mathrm{T}}\mathbf{p}. \end{aligned} \tag{77}$$

As soon as $\mathbf{c}_k$ are known, $\mathbf{f}_1(\zeta_*)$ can be obtained from eqns $(72)_1$ or $(73)_1$. For convenience, $\mathbf{f}_0(\zeta_*)$, $\mathbf{f}_1(\zeta_*)$ and $\mathbf{f}_2(\zeta_*)$ in eqns (65) and (66) are listed here, in which a replacement from symbolical expression to matrix notation in $\mathbf{f}_1(\zeta_*)$ and $\mathbf{f}_2(\hat{\zeta}_*)$ has been made:

$$\mathbf{f}_0(\zeta_*) = \frac{1}{2\pi i}\langle \ln(\zeta_\alpha - \zeta_\alpha^*)\rangle \mathbf{A}_1^{\mathrm{T}}\mathbf{p}, \quad \zeta_\alpha \in S_1,$$

$$\mathbf{f}_1(\zeta_*) = \sum_{k=1}^{\infty} \mathbf{B}_1^{-1}\langle \zeta_\alpha^{-k}\rangle(\bar{\mathbf{B}}_2\bar{\mathbf{c}}_k + \mathbf{B}_2\hat{\mathbf{\Gamma}}_k\mathbf{c}_k) - \mathbf{B}_1^{-1}\bar{\mathbf{B}}_1\bar{\mathbf{f}}_0(1/\zeta_*), \quad \zeta_\alpha \in S_1,$$

$$\mathbf{f}_2(\hat{\zeta}_*) = \sum_{k=1}^{\infty} \{\langle \hat{\zeta}_\alpha^k\rangle \mathbf{c}_k + \langle \hat{\zeta}_\alpha^{-k}\rangle \mathbf{\Gamma}_k\mathbf{c}_k\}, \quad \hat{\zeta}_\alpha \in S_2 \quad \text{and} \quad |\hat{\zeta}_\alpha| < 1, \tag{78}$$

where $\mathbf{c}_k$ are given from eqns (75)–(77), and $\langle \zeta_\alpha^{-k}\rangle$, $\langle \hat{\zeta}_\alpha^k\rangle$ and $\langle \hat{\zeta}_\alpha^{-k}\rangle$ are all four-order diagonal matrices. With the solutions, stresses and electric displacements in the matrix and the inclusion and on the interface boundary could be determined similarly following the way given in the last section. When the piezoelectric inclusion is replaced by a traction-free hole, the solution in this section can be reduced to the results of the last section.

5. CONCLUSIONS

The Green functions of an infinite two-dimensional piezoelectric material containing an elliptic hole or an elliptic inclusion are obtained. The results are of importance because they could be used as kernels of boundary integral equations in BEM analyses. Since the Green functions have satisfied the boundary conditions of hole or inclusion interface, the boundary integrals along these surfaces can then be avoided. This property is very useful especially in numerical approach.

It is pointed out here again that Stroh formalism for anisotropy elasticity could be thought as a special case of extended Stroh formalism for piezoelectric material. Therefore, many useful results for anisotropic elasticity could be extended to similar problems for piezoelectric materials with certain modifications. The present work is also an extension of anisotropic elasticity work (Hwu and Yen, 1991; Yen and Hwu, 1994), and it is hopeful that the result could have certain promotion to the studies concerning numerical computations for piezoelectric problems.

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